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# On a class of non-linear delay distributed order fractional diffusion equations

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ABSTRACT

In this paper, we consider a numerical scheme for a class of non-linear time delay fractional diffusion equations with distributed order in time. This study covers the unique solvability, convergence and stability of the resulted numerical solution by means of the discrete energy method. The derivation of a linearized difference scheme with convergence order $O(\tau + (\Delta\alpha)^4 + h^4)$ in $L_\infty$-norm is the main purpose of this study. Numerical experiments are carried out to support the obtained theoretical results.



## 1. Introduction

In the past few decades, high and rapid growing attention related with partial differential equations which contain fractional derivatives and integrals occurred. The ability of the models which contain non-integer orders comparing with integers order models in describing some certain phenomena is more accurate. The need of many scientific areas for the use of fractional partial differential equations (FPDEs) to describe their processes has been widely recognized. Nowadays, the interest of scientists with FPDEs in fields of finance [1], engineering [2], viscoelasticity [3], control systems [4], diffusion procedures [5] and many other scientific areas has no limit. Many anomalous diffusion processes which existed in some physical and biological areas can be modeled by the time fractional reaction diffusion wave equation [6,7]. Recently, distributed order fractional differential equations can model perfectly some different problems in mathematical physics and engineering [8,9]. As one of the realistic models of these equations, the authors in [10] transfer the multi-term fractional derivative viscoelastic model to a derivative model of distributed order and checked its effect on several systems such as the fractional distributed order oscillator and the distributed order fractional wave equation. Also, in [11] a distributed-order fractional diffusion-wave equation is used to describe radial ground water flow to or from a well, and three sets of solutions are obtained for flow from a well for aquifer tests: one for pumping tests, and two for slug tests. In the simulation of dynamical systems, two effects (distribution of parameters in space and delay in time) often existed. Due to that, we study the effect of entering a delay term in the source function of distributed order fractional diffusion equations. The existence

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and the uniqueness of mild and classical solutions for a class of distributed order fractional differential equations had been studied in [12]. Some authors extended the multi-term fractional derivative models to models with distributed orders and as a result of these generalizations, some new systems such as the distributed order fractional wave equation [13] and fractional distributed order oscillator [14] were presented. A fundamental solution of a distributed order time fractional diffusion wave equation appeared in [15]. The time fractional diffusion wave equation has the following form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = K\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t)), \quad t>0,\ 0\le x\le L, \tag{1}$$

with the fractional order $\alpha \in (0,1]$; if $\alpha = 1$, the classical diffusion equation is recovered. The numerical solutions of such equations were proposed in the literature by means of finite difference methods [16,17], spectral collocation methods [18,19] and others. In the literature, a considerable attention to deal with the general class (distributed order form) of (1) is discussed in [20,21]. The distributed order in time fractional diffusion equation can be written in the following form

$$\int_0^1 \omega(\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}d\alpha = K\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t)), \quad t>0,\ 0\le x\le L. \tag{2}$$

The demand of obtaining numerical solutions, such as in [22,23,21,24], to treat these equations aims to overcome the mathematical complexity of analytical solutions. For the two dimensional form of (2), the authors in [25,26] derived high order difference schemes. Also, time delay differential equations are widely used in many fields (economics, medicine, physics, etc.) [27,28]. The theory of delay differential equations has a great interest and is developing rapidly [29]. We are concerned with a generalization of (2) to include a non-linear delayed source function, more specific we consider

$$\int_0^1 \omega(\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}d\alpha = K\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t),u(x,t-s)), \quad t>0,\ 0\le x\le L, \tag{3a}$$

with the following initial and boundary conditions

$$u(x,t) = \psi(x,t), \quad 0\le x\le L, \quad t\in[-s,0), \tag{3b}$$

$$u(0,t) = \phi_0(t), \quad u(L,t)=\phi_L(t), \quad t>0, \tag{3c}$$

where $s>0$ is the delay parameter, $K$ is a positive constant, and $\omega(\alpha)>0$ is a weight function. The fractional derivative is introduced in Caputo sense, that is

$${}_0^C D_t^\alpha u(x,t) \equiv \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} := \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\zeta)^{-\alpha}\frac{\partial u(x,\zeta)}{\partial \zeta}d\zeta, \quad 0<\alpha<1. \tag{4}$$

We have two degrees of complexity, the distributed order and the nonlinear delay source function. We seek to obtain a numerical solution for (3). Throughout this work, like in [30], we suppose that the function $f(x,t,\mu,\nu)$ and the solution $u(x,t)$ of (3) are sufficiently smooth in the following sense:

- Let $m$ be an integer satisfying $ms\le T<(m+1)s$, define $I_r = (rs,(r+1)s)$, $r=-1,0,\dots,m-1$, $I_m=(ms,T)$, $I=\bigcup_{q=-1}^m I_q$ and assume that $u(x,t)\in C^{(6,4)}([0,L]\times(0,T])$,
- The partial derivatives $f_\mu(x,t,\mu,\nu)$ and $f_\nu(x,t,\mu,\nu)$ are continuous in the $\epsilon_0$-neighborhood of the solution. Define

$$c_1 = \sup_{\substack{0<x<L,0<t\le T\\ |\epsilon_1|\le\epsilon_0,|\epsilon_2|\le\epsilon_0}} \left|f_\mu(x,t,u(x,t)+\epsilon_1,u(x,t-s)+\epsilon_2)\right|,$$

$$c_2 = \max_{\substack{0<x<L,0<t\le T\\ |\epsilon_1|\le\epsilon_0,|\epsilon_2|\le\epsilon_0}} \left|f_\nu(x,t,u(x,t)+\epsilon_1,u(x,t-s)+\epsilon_2)\right|.$$

Difference schemes can be applied to solve numerically and study some different sorts of differential equations [31–34]. Numerical studies for fractional functional differential equations with delay based on BDF-type shifted Chebyshev polynomials are exhibited in [35]. In [36], Ferreira introduced energy estimates for delay diffusion–reaction and studied the following nonlinear delay partial differential equations

$$\frac{\partial u}{\partial t} - \alpha\frac{\partial^2 u}{\partial x^2} = f(u(x,t),u(x,t-s)), \quad a<x<b,\ t\in[0,T], \tag{5a}$$

$$u(a,t) = u_a(t), \quad u(b,t)=u_b(t),\ t\in[0,T], \tag{5b}$$

$$u(x,t) = \rho(x,t), \quad x\in[a,b],\ t\in[-s,0). \tag{5c}$$

The author gave a backward Euler scheme with $L_2$-convergence order $O\left(\tau+h^2\right)$. Zhang and Sun [37] introduced a linearized compact difference scheme for a class of nonlinear delay partial differential equations with initial and Dirichlet boundary conditions. The unique solvability, unconditional convergence and stability of the scheme are proved. The convergence order is $O\left(\tau^2+h^4\right)$ in the $L_\infty$-norm. In [38], Karatay and his group obtained an approximation for the time Caputo fractional derivative at time $t_{k+1/2}$ with fractional order $0 < \alpha < 1$ and they extend the idea of the Crank–Nicolson method to time fractional heat equations with convergence order $O\left(\tau^{2-\alpha} + h^2\right)$. Based on the ideas in [38,37], we construct a linearized difference scheme for (3). The structure of this paper is arranged in the following way: we present the derivation of the difference scheme in the following section. Next, in Section 3, the solvability, convergence and stability for the difference scheme are discussed. In Section 4, numerical examples are given to illustrate the accuracy of the presented scheme and to support our theoretical results. Finally, the paper ends with a conclusion and some remarks.

## 2. Derivation of the difference scheme

We seek to obtain a numerical solution based on the Crank–Nicolson method. First we transform the distributed order diffusion equation (3a) into a multi-term fractional differential equation with delay by using a numerical quadrature formula. We briefly recall Simpson's rule (also known as the three-point Newton–Cotes quadrature rule).

**Lemma 2.1.** *Consider an equidistant partition of the interval* $[0, 1]$ *into* $2J$ *subintervals, let* $\Delta\alpha = \frac{1}{2J}$ *and denote* $\alpha_l = l\,\Delta\alpha$, $0 \le l \le 2J$. *Then, the composite Simpson's rule reads*

$$\int_0^1 f(\alpha)d\alpha = \Delta\alpha \sum_{l=0}^{2J} \gamma_l f(\alpha_l) - \frac{(\Delta\alpha)^4}{180} f^{(4)}(\zeta), \quad \zeta \in [0, 1], \tag{6}$$

*where*

$$\gamma_l = \begin{cases} \dfrac{1}{3}, & l = 0, 2J, \\ \dfrac{2}{3}, & l = 2, 4, \ldots, 2J-4, 2J-2, \\ \dfrac{4}{3}, & l = 1, 3, \ldots, 2J-3, 2J-1. \end{cases}$$

We fix some further notations. Take two positive integers $M$ and $n$, let $h = \frac{L}{M}$, $\tau = \frac{s}{n}$ and denote $x_i = ih$, $t_k = k\tau$ and $t_{k+1/2} = \left(k + \frac{1}{2}\right)\tau = \frac{1}{2}(t_k + t_{k+1})$. Cover the space–time domain by $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, where $\Omega_h = \{x_i \mid 0 \le i \le M\}$, $\Omega_\tau = \{t_k \mid -n \le k \le N\}$, $N = \left\lfloor \frac{T}{\tau} \right\rfloor$. Let $\mathcal{W} = \{v \mid v = v_i^k, 0 \le i \le M, -n \le k \le N\}$ be a grid function space on $\Omega_{h\tau}$. For $v \in \mathcal{W}$ we denote $v_i^{k+1/2} = \frac{1}{2}\left(v_i^k + v_i^{k+1}\right)$ and $\delta_x^2 v_i^k = \frac{1}{h^2}\left(v_{i+1}^k - 2v_i^k + v_{i-1}^k\right)$.

**Lemma 2.2** ([37]). *Let* $q(x) \in C^6[x_{i-1}, x_{i+1}]$, *then*

$$\frac{1}{12}\left(q''(x_{i-1}) + 10q''(x_i) + q''(x_{i+1})\right) - \frac{1}{h^2}\left(q(x_{i-1}) - 2q(x_i) + q(x_{i+1})\right) = \frac{h^4}{240} q^{(6)}(\omega_i),$$

*where* $\omega_i \in (x_{i-1}, x_{i+1})$.

Define the function $G(\alpha) = \omega(\alpha)\frac{\partial^\alpha u}{\partial t^\alpha}$, $\alpha \in (0, 1]$. Suppose that $G(\alpha) \in C^4[0, 1]$, then using Lemma 2.1, we approximate the distributed integral as

$$\begin{aligned} \int_0^1 \omega(\alpha)\frac{\partial^\alpha u(x_i, t_{k+1/2})}{\partial t^\alpha} d\alpha &= \Delta\alpha \sum_{l=0}^{2J} \gamma_l \omega(\alpha_l)\, {}_0^C D_t^{\alpha_l} u(x_i, t_{k+1/2}) - \frac{(\Delta\alpha)^4}{180} \Phi^{(4)}(\alpha; x_i, t_{k+1/2})\Big|_{\alpha=\zeta_i^k}, \\ &= \Delta\alpha \sum_{l=0}^{2J} \gamma_l \omega(\alpha_l)\, {}_0^C D_t^{\alpha_l} u(x_i, t_{k+1/2}) + O\left(\Delta\alpha\right)^4, \end{aligned} \tag{7}$$

where $\zeta_i^k \in [0, 1]$, and we defined $\Phi(\alpha; x_i, t_{k+1/2}) = \omega(\alpha)\frac{\partial^\alpha u(x_i, t_{k+1/2})}{\partial t^\alpha}$.

We define the grid function on $\Omega_{h\tau}$: $U(i, k) = u(x_i, t_k)$. In [38] an approximation for the time Caputo fractional derivative at $t_{k+1/2}$ with $0 < \alpha_l < 1$ was given:

$$\frac{\partial^{\alpha_l} u(x_i, t_{k+1/2})}{\partial t^{\alpha_l}} = \omega_1^l U_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1}^l - \omega_{k-m}^l) u_i^m - \omega_k^l U_i^0 + \frac{\sigma^l}{2^{1-\alpha_l}}\left(U_i^{k+1} - U_i^k\right) + O\left(\tau^{2-\alpha_l}\right), \tag{8}$$

where

$$\omega_i^l = \sigma^l\left(\left(i + \frac{1}{2}\right)^{1-\alpha_l} - \left(i - \frac{1}{2}\right)^{1-\alpha_l}\right), \quad \sigma^l = \frac{1}{\tau^{\alpha_l}\Gamma(2 - \alpha_l)}, \; 0 \le i \le M, \; 0 \le k \le N-1. \tag{9}$$

We are now in a position to apply and combine the above, that is (7) and (8), to (3a) at the points $(x_i, t_{k+1/2})$, and arrive at

$$\Delta\alpha \sum_{l=0}^{2J} \gamma_l \omega(\alpha_l) \left[ \omega_1^l U_i^k + \sum_{m=1}^{k-1} (\omega_{k-m+1}^l - \omega_{k-m}^l) U_i^m - \omega_k^l U_i^0 + \frac{\sigma^l}{2^{1-\alpha_l}} \left(U_i^{k+1} - U_i^k\right) + O\left(\tau^{2-\alpha_l}\right) \right] + O\left(\Delta\alpha\right)^4$$
$$= K \frac{\partial^2 u(x_i, t_{k+1/2})}{\partial x^2} + f(x_i, t_{k+1/2}, u(x_i, t_{k+1/2}), u(x_i, t_{k+1/2} - s)). \tag{10}$$

**Lemma 2.3.** *For $g = (g_0, g_1, \dots, g_M)$, let the linear operator $\mathfrak{A}$ is defined as*

$$\mathfrak{A} g_i = \frac{1}{12}(g_{i-1} + 10 g_i + g_{i+1}), \quad 1 \le i \le M-1.$$

*Then, we obtain*

$$\Delta\alpha \sum_{l=0}^{2J} \gamma_l \omega(\alpha_l) \mathfrak{A} \left[ \omega_1^l U_i^k + \sum_{m=1}^{k-1} (\omega_{k-m+1}^l - \omega_{k-m}^l) U_i^m - \omega_k^l U_i^0 + \frac{\sigma^l}{2^{1-\alpha_l}} \left(U_i^{k+1} - U_i^k\right) \right]$$
$$= K \delta_x^2 U_i^{k+1/2} + \mathfrak{A} f\left(x_i, t_{k+1/2}, \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}\right) + R_i^k, \tag{11}$$

*where*

$$\left|R_i^k\right| = O\left((\Delta\alpha)^4 + h^4 + \tau\right), \quad 1 \le i \le M-1, \ 0 \le k \le N-1. \tag{12}$$

**Proof.** We use the Taylor expansions

$$\frac{\partial^2 u(x_i, t_{k+1/2})}{\partial x^2} = \frac{1}{2}\left(\frac{\partial^2 u(x_i, t_k)}{\partial x^2} + \frac{\partial^2 u(x_i, t_{k+1})}{\partial x^2}\right) + O\left(\tau^2\right),$$
$$u(x_i, t_{k+1/2}) = U_i^{k+1/2} = \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1} + O\left(\tau^2\right),$$
$$u(x_i, t_{k+1/2} - s) = U_i^{k-n+\frac{1}{2}} = \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n} + O\left(\tau^2\right),$$

in (10) and obtain

$$\Delta\alpha \sum_{l=0}^{2J} \gamma_l \omega(\alpha_l) \left[ \omega_1^l U_i^k + \sum_{m=1}^{k-1} (\omega_{k-m+1}^l - \omega_{k-m}^l) U_i^m - \omega_k^l U_i^0 + \frac{\sigma^l}{2^{1-\alpha_l}} \left(U_i^{k+1} - U_i^k\right) + O\left(\tau^{2-\alpha_l}\right) \right] + O\left(\Delta\alpha\right)^4$$
$$= \frac{K}{2}\left(\frac{\partial^2 u(x_i, t_k)}{\partial x^2} + \frac{\partial^2 u(x_i, t_{k+1})}{\partial x^2}\right) + O\left(\tau^2\right) + f\left(x_i, t_{k+1/2}, \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}\right),$$

where we used the continuity of the derivatives of $f$ in its third and fourth components. We rewrite this as

$$\Delta\alpha \sum_{l=0}^{2J} \gamma_l \omega(\alpha_l) \left[ \omega_1^l U_i^k + \sum_{m=1}^{k-1} (\omega_{k-m+1}^l - \omega_{k-m}^l) U_i^m - \omega_k^l U_i^0 + \frac{\sigma^l}{2^{1-\alpha_l}} \left(U_i^{k+1} - U_i^k\right) \right] + O(\tau)$$
$$= \frac{K}{2}\left(\frac{\partial^2 u(x_i, t_k)}{\partial x^2} + \frac{\partial^2 u(x_i, t_{k+1})}{\partial x^2}\right) + f\left(x_i, t_{k+1/2}, \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}\right) + O\left(\tau^2 + (\Delta\alpha)^4\right), \tag{13}$$

because $\Delta\alpha \sum_{l=0}^{2J} \gamma_l \, \omega(\alpha_l) \, O\left(\tau^{2-\alpha_l}\right) = O(\tau) \, \Delta\alpha \sum_{l=0}^{2J} \gamma_l \, \omega(\alpha_l)$ and $\Delta\alpha \sum_{l=0}^{2J} \gamma_l \, \omega(\alpha_l)$ is bounded by Lemma 2.1. According to Lemma 2.2 we have

$$\mathfrak{A} \frac{\partial^2 u(x_i, t_k)}{\partial x^2} = \delta_x^2 U_i^k + \frac{h^4}{240} \frac{\partial^6 u}{\partial x^6}(\theta_i^k, t_k), \quad \theta_i^k \in (x_{i-1}, x_{i+1}),$$

so applying $\mathfrak{A}$ to (13) we arrive at

$$\Delta\alpha \sum_{l=0}^{2J} \gamma_l \omega(\alpha_l) \mathfrak{A} \left[ \omega_1^l U_i^k + \sum_{m=1}^{k-1} (\omega_{k-m+1}^l - \omega_{k-m}^l) u_i^m - \omega_k^l U_i^0 + \frac{\sigma^l}{2^{1-\alpha_l}} \left(U_i^{k+1} - U_i^k\right) \right]$$
$$= K \delta_x^2 U_i^{k+1/2} + \mathfrak{A} f\left(x_i, t_{k+1/2}, \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}\right) + O\left(\tau + (\Delta\alpha)^4 + h^4\right)$$

as $u(x, t) \in C^{(6,4)}(I \times (0, T])$. □

The final form of our difference scheme is obtained by neglecting $R_i^k$ and replace $U_i^k$ with $u_i^k$ in (11)

$$\Delta\alpha \sum_{l=0}^{2J} \gamma_l \omega(\alpha_l)\mathfrak{A}\left[\omega_1^l u_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1}^l - \omega_{k-m}^l)u_i^m - \omega_k^l u_i^0 + \frac{\sigma^l}{2^{1-\alpha_l}}\left(u_i^{k+1} - u_i^k\right)\right]$$
$$= K\delta_x^2 u_i^{k+1/2} + \mathfrak{A}f\left(x_i, t_{k+1/2}, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}\right), \tag{14a}$$

and supplying appropriate initial and boundary conditions

$$u_0^k = \phi_0(t_k), \quad u_M^k = \phi_L(t_k), \ 1 \le k \le N, \tag{14b}$$

$$u_i^k = \psi(x_i, t_k), \quad 0 \le i \le M, \quad -n \le k \le 0. \tag{14c}$$

## 3. The solvability, convergence and stability for the difference scheme

Now, we introduce the uniqueness, stability and convergence theorems in $L_\infty$ norm using the discrete energy method for the proposed difference scheme.

If the spatial domain $[0, L]$ is covered by $\Omega_h = \{x_i \mid 0 \le i \le M, \}$ and let $V_h = \{v \mid v = (v_0, \dots, v_M), v_0 = v_M = 0\}$ be a grid function space on $\Omega_h$. For any $u, v \in V_h$, define the discrete inner products and corresponding norms as

$$\langle u, v\rangle = h\sum_{i=1}^{M-1} u_i v_i, \qquad \langle \delta_x u, \delta_x v\rangle = h\sum_{i=1}^{M}(\delta_x u_{i-1/2})(\delta_x v_{i-1/2}), \qquad \langle \delta_x^2 u, v\rangle = -\langle \delta_x u, \delta_x v\rangle, \qquad \delta_x u_i = \frac{1}{h}(u_i - u_{i-1})$$

and

$$\|u\| = \sqrt{\langle u, u\rangle}, \qquad |u|_1 = \sqrt{\langle \delta_x u, \delta_x u\rangle}, \qquad \|u\|_\infty = \max_{0\le i\le M} |u|.$$

According to [37], the following inequalities are fulfilled

$$\|u\|_\infty \le \frac{\sqrt{L}}{2}|u|_1, \qquad \|u\| \le \frac{L}{\sqrt{6}}|u|_1. \tag{15}$$

For the analysis of the difference scheme, we need to use the following inequality:

**Lemma 3.1** (*Gronwall Inequality [37]*). *Suppose that $\{F^k \mid k \ge 0\}$ is a nonnegative sequence and satisfies $F^{k+1} \le A + B\tau\sum_{l=1}^{k} F^l$, $k \ge 0$, for some nonnegative constants $A$ and $B$. Then $F^{k+1} \le A\exp(Bk\tau)$.*

We now prove that our difference scheme admits a unique solution. Next we show that the obtained solution solves (3).

**Theorem 1.** *The difference scheme (14) is uniquely solvable.*

**Proof.** Suppose that $u_i^k$, $0 \le i \le M$ is the solution for the obtained difference scheme (14). Using the mathematical induction, the base step is fulfilled from the initial condition (14c) as the solution $u_i^k$ is determined for $-n \le k \le 0$. For the inductive hypothesis, let $u_i^k$ be determined when $k = l$, then from (14a) we obtain a system of linear algebraic equations with respect to $u_i^l$. The proof ends by the inductive step as the coefficient matrix of this system is strictly diagonally dominant, so there exists a unique solution $u_i^{l+1}$. We can arrange the system (14) as follows

$$\left(\left[\frac{\Delta\alpha}{12}\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\frac{\sigma^l}{2^{1-\alpha_l}} - \frac{K}{2h^2}\right]u_{i+1}^{k+1} + \left[\frac{10\Delta\alpha}{12}\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\frac{\sigma^l}{2^{1-\alpha_l}} + \frac{K}{h^2}\right]u_i^{k+1}\right.$$
$$\left. + \left[\frac{\Delta\alpha}{12}\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\frac{\sigma^l}{2^{1-\alpha_l}} - \frac{K}{2h^2}\right]u_{i-1}^{k+1}\right) + \left(\left[\frac{\Delta\alpha}{12}\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\left(\omega_1^l - \frac{\sigma^l}{2^{1-\alpha_l}}\right) - \frac{K}{2h^2}\right]u_{i+1}^k\right.$$
$$\left. + \left[\frac{10\Delta\alpha}{12}\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\left(\omega_1^l - \frac{\sigma^l}{2^{1-\alpha_l}}\right) + \frac{K}{h^2}\right]u_i^k + \left[\frac{\Delta\alpha}{12}\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\left(\omega_1^l - \frac{\sigma^l}{2^{1-\alpha_l}}\right) - \frac{K}{2h^2}\right]u_{i-1}^k\right)$$
$$+ \mathfrak{A}\left(\sum_{m=1}^{k-1}(\omega_{k-m+1}^l - \omega_{k-m}^l)u_i^m - \omega_k^l u_i^0\right) = \mathfrak{A}f\left(x_i, t_{k+1/2}, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}\right).$$

According to the system above, the coefficient matrix $\mathsf{A}=(a_{ij})$ is strictly diagonally dominant because $a_{ii}\geq \sum_{j\neq i}|a_{ij}|$;

$$a_{ii}=\frac{10}{12}\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\frac{\sigma^l\Delta\alpha}{2^{1-\alpha_l}}+\frac{K}{h^2},\qquad a_{i+1,i}=\frac{1}{12}\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\frac{\sigma^l\Delta\alpha}{2^{1-\alpha_l}}-\frac{K}{2h^2}=a_{i-1,i},\qquad \frac{\sigma^l}{2^{1-\alpha}}>0.$$

Therefore, the coefficient matrix is nonsingular and this proves the theorem. □

**Theorem 2** (*Convergence Theorem*). *Let $u(x,t)\in[0,L]\times[-s,T]$, be the solution of (3) such that $u(x_i,t_k)=U_i^k$ and $u_i^k$ $(0\leq i\leq M,\,-n\leq k\leq N)$ be the solution of the difference scheme (14), denote $e_i^k=U_i^k-u_i^k$, for $0\leq i\leq M$, $-n\leq k\leq N$, and*

$$C=\frac{3c_3L}{2}\sqrt{\frac{T}{5K\Delta\alpha\omega(1)}}\exp\left(\frac{3L^2\left(c_1^2+\frac{c_2^2}{5}\right)T}{4K\Delta\alpha\omega(1)}\right),$$

*then if*

$$\tau\leq\tau_0=\left(\frac{\epsilon_0}{6C}\right),\qquad h\leq h_0=\left(\frac{\epsilon_0}{6C}\right)^{\frac{1}{4}},\qquad \Delta\alpha\leq\left(\frac{\epsilon_0}{6C}\right)^{\frac{1}{4}},\tag{16}$$

*one has that*

$$\|e^k\|_\infty\leq C\left(\tau+(\Delta\alpha)^4+h^4\right),\quad 0\leq k\leq N.\tag{17}$$

**Proof.** The error difference scheme can be obtained by subtracting (14a) from (11), the latter with $u$ replaced by $U$, as follows

$$\begin{aligned}\Delta\alpha\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\mathfrak{A}&\left[\omega_1^le_i^k+\sum_{m=1}^{k-1}(\omega_{k-m+1}^l-\omega_{k-m}^l)e_i^m-\omega_k^le_i^0+\frac{\sigma^l}{2^{1-\alpha_l}}\left(e_i^{k+1}-e_i^k\right)\right]=K\delta_x^2e_i^{k+1/2}+R_i^k\\&+\mathfrak{A}\left[f\left(x_i,t_{k+1/2},\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\right)\right.\\&\left.-f\left(x_i,t_{k+1/2},\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\right)\right],\end{aligned}\tag{18}$$

and

$$e_0^k=0,\qquad e_M^k=0,\quad 1\leq k\leq N,\tag{19}$$

$$e_i^k=0,\quad 0\leq i\leq M,\quad -n\leq k\leq 0.\tag{20}$$

By taking the inner product of each part of (18) with $\delta_te_i^{k+1/2}$, this yields

$$\begin{aligned}\Delta\alpha\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)&\left\langle\mathfrak{A}\left[\omega_1^le_i^k+\sum_{m=1}^{k-1}(\omega_{k-m+1}^l-\omega_{k-m}^l)e_i^m-\omega_k^le_i^0+\frac{\sigma^l}{2^{1-\alpha_l}}\left(e_i^{k+1}-e_i^k\right)\right],\delta_te_i^{k+1/2}\right\rangle\\&=K\left\langle\delta_x^2e_i^{k+1/2},\delta_te_i^{k+1/2}\right\rangle+\left\langle R_i^k,\delta_te_i^{k+1/2}\right\rangle+\left\langle\mathfrak{A}\left[f\left(x_i,t_{k+1/2},\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\right)\right.\right.\\&\left.\left.-f\left(x_i,t_{k+1/2},\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\right)\right],\delta_te_i^{k+1/2}\right\rangle.\end{aligned}\tag{21}$$

We will prove (17) by strong mathematical induction. The base case is evident; following (20), it is clear that $\|e^k\|_\infty=0,\;-n\leq k\leq 0$, so in particular we have $\|e^0\|_\infty=0$. Next, suppose that (17) is fulfilled for $0\leq k\leq\ell$, then we will show that (17) holds for $k=\ell+1$.

From the inductive hypothesis, and when $\tau$ and $h$ satisfy (16), we obtain

$$\|e^k\|_\infty\leq C\left(\tau+(\Delta\alpha)^4+h^4\right)\leq\epsilon_0/2,\quad 0\leq k\leq\ell.\tag{22}$$

From (22), we conclude that $|e^k|\leq\epsilon_0/2$, $0\leq k\leq\ell$, and so $|U_i^k-u_i^k|\leq\epsilon_0/2$, $|U_i^{k-1}-u_i^{k-1}|\leq\epsilon_0/2$, $0\leq k\leq\ell$, then $|\frac{3}{2}(U_i^k-u_i^k)-\frac{1}{2}(U_i^{k-1}-u_i^{k-1})|\leq\epsilon_0/2$, then the following inequality is fulfilled

$$\left|\left(\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1}\right)-\left(\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1}\right)\right|\leq\epsilon_0,\quad 0\leq i\leq M,\;0\leq k\leq\ell.$$

By the same way, we conclude that $|\frac{1}{2}(U_i^{k+1-n}-u_i^{k+1-n})+\frac{1}{2}(U_i^{k-n}-u_i^{k-n})|\le\epsilon_0/2$, then the following inequality is achieved

$$\left|\left(\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\right)-\left(\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\right)\right|\le\epsilon_0,\quad 0\le i\le M,\ 0\le k\le\ell.$$

Consequently,

$$\begin{aligned}&\left|f\left(x_i,t_{k+1/2},\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\right)-f\left(x_i,t_{k+1/2},\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\right)\right|\\&\quad\le c_1\left|\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}\right|+c_2\left|\frac{1}{2}e_i^{k+1-n}+\frac{1}{2}e_i^{k-n}\right|,\quad 0\le i\le M,\ 0\le k\le\ell,\end{aligned}$$

and so

$$\begin{aligned}&\left|\mathfrak{A}\left[f\left(x_i,t_{k+1/2},\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\right)-f\left(x_i,t_{k+1/2},\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\right)\right]\right|\\&\quad\le\mathfrak{A}\left(c_1\left|\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}\right|+c_2\left|\frac{1}{2}e_i^{k+1-n}+\frac{1}{2}e_i^{k-n}\right|\right),\quad 0\le i\le M,\ 0\le k\le\ell,\end{aligned}\tag{23}$$

using (23), we can predict that

$$\begin{aligned}&\left\langle\mathfrak{A}\left[f\left(x_i,t_{k+1/2},\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\right)-f\left(x_i,t_{k+1/2},\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\right)\right],\delta_te_i^{k+\frac{1}{2}}\right\rangle\\&\quad\le\left\langle\mathfrak{A}\left(c_1\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}\mid+c_2\mid\frac{1}{2}e_i^{k+1-n}+\frac{1}{2}e_i^{k-n}\right),\delta_te_i^{k+\frac{1}{2}}\right\rangle.\end{aligned}\tag{24}$$

For simplicity, the inner product in the r.h.s of (24) will be denoted by $\langle\xi_1,\xi_2\rangle$. The use of the holder inequality gives $\langle\xi_1,\xi_2\rangle\le\frac{1}{2\theta}\|\xi_1\|^2+\frac{\theta}{2}\|\xi_2\|^2$, and by taking $\theta=\frac{5\Delta\alpha\omega(1)}{18}$, we obtain

$$R.H.s\le\frac{18}{10\Delta\alpha\omega(1)}\left\|\mathfrak{A}\left(c_1\left|\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}\right|+c_2\left|\frac{1}{2}e_i^{k+1-n}+\frac{1}{2}e_i^{k-n}\right|\right)\right\|^2+\frac{5\Delta\alpha\omega(1)}{36}\left\|\delta_te_i^{k+\frac{1}{2}}\right\|^2,\tag{25}$$

also,

$$\langle\mathfrak{A}g,g\rangle\le(g,g),\quad\forall g\in V,$$

then

$$\begin{aligned}R.H.s&\le\frac{18}{10\Delta\alpha\omega(1)}\left\|c_1\left|\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}\right|+c_2\left|\frac{1}{2}e_i^{k+1-n}+\frac{1}{2}e_i^{k-n}\right|\right\|^2+\frac{5\Delta\alpha\omega(1)}{36}\left\|\delta_te_i^{k+\frac{1}{2}}\right\|^2, &(26)\\&=\frac{18}{10\Delta\alpha\omega(1)}h\sum_{i=1}^{M-1}\left(c_1\left|\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}\right|+c_2\left|\frac{1}{2}e_i^{k+1-n}+\frac{1}{2}e_i^{k-n}\right|\right)^2+\frac{5\Delta\alpha\omega(1)}{36}\left\|\delta_te_i^{k+\frac{1}{2}}\right\|^2, &(27)\\&\le\frac{18}{10\Delta\alpha\omega(1)}\left[hc_1^2\sum_{i=1}^{M-1}\left(\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}\right)^2+c_2^2h\sum_{i=1}^{M-1}\left(\frac{1}{2}e_i^{k+1-n}+\frac{1}{2}e_i^{k-n}\right)^2\right]+\frac{5\Delta\alpha\omega(1)}{36}\left\|\delta_te_i^{k+\frac{1}{2}}\right\|^2, &(28)\\&\le\frac{18}{10\Delta\alpha\omega(1)}\left[\frac{5}{2}hc_1^2\sum_{i=1}^{M-1}\left((e_i^k)^2+(e_i^{k-1})^2\right)+\frac{1}{2}c_2^2h\sum_{i=1}^{M-1}\left((e_i^{k+1-n})^2+(e_i^{k-n})^2\right)\right]+\frac{5\Delta\alpha\omega(1)}{36}\left\|\delta_te_i^{k+\frac{1}{2}}\right\|^2, &(29)\\L.H.s&\le\frac{9}{\Delta\alpha\omega(1)}c_1^2\left(\|e_i^k\|^2+\|e_i^{k-1}\|^2\right)+\frac{9}{5\Delta\alpha\omega(1)}c_2^2\left(\|e_i^{k+1-n}\|^2+\|e_i^{k-n}\|^2\right)+\frac{5\Delta\alpha\omega(1)}{36}\left\|\delta_te_i^{k+\frac{1}{2}}\right\|^2. &(30)\end{aligned}$$

Also, for the second inner product in the r.h.s of (21) and using Holder inequality in the same way as before, we obtain

$$\left\langle R_i^k,\delta_te_i^{k+1/2}\right\rangle\le\frac{18}{10\Delta\alpha\omega(1)}\|R_i^k\|^2+\frac{5\Delta\alpha\omega(1)}{36}\left\|\delta_te_i^{k+\frac{1}{2}}\right\|^2.\tag{31}$$

For the first inner product in the r.h.s of (21),

$$K\left\langle\delta_x^2e_i^{k+1/2},\delta_te_i^{k+1/2}\right\rangle=\frac{-K}{2\tau}(|e_i^{k+1}|_1^2-|e_i^k|_1^2).\tag{32}$$

As, see (9), we have

$$\sum_{m=1}^{k-1}(\omega_{k-m+1}^l-\omega_{k-m}^l)=\omega_k^l-\omega_1^l>0,\quad \omega_1^l>0\Rightarrow\omega_k^l>0,\ k>1. \tag{33}$$

With the aid of (33), we have the following estimate for the l.h.s. of (21),

$$\begin{aligned}
&\Delta\alpha\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\left\langle\mathfrak{A}\left[\omega_1^l e_i^k+\sum_{m=1}^{k-1}(\omega_{k-m+1}^l-\omega_{k-m}^l)e_i^m-\omega_k^l e_i^0+\frac{\sigma^l}{2^{1-\alpha_l}}\left(e_i^{k+1}-e_i^k\right)\right],\delta_t e_i^{k+1/2}\right\rangle\\
&\geq\Delta\alpha\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\left\langle\mathfrak{A}\left[\omega_1^l e_i^0+\sum_{m=1}^{k-1}(\omega_{k-m+1}^l-\omega_{k-m}^l)e_i^0-\omega_k^l e_i^0+\frac{\sigma^l}{2^{1-\alpha_l}}\left(e_i^{k+1}-e_i^k\right)\right],\delta_t e_i^{k+1/2}\right\rangle\\
&=\Delta\alpha\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\left\langle\mathfrak{A}\left[\frac{\sigma^l}{2^{1-\alpha_l}}\left(e_i^{k+1}-e_i^k\right)\right],\delta_t e_i^{k+1/2}\right\rangle\\
&=\Delta\alpha\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\frac{\tau^{1-\alpha_l}}{2^{1-\alpha_l}\Gamma(2-\alpha_l)}\left\langle\mathfrak{A}\left[\frac{e_i^{k+1}-e_i^k}{\tau}\right],\delta_t e_i^{k+1/2}\right\rangle\\
&\geq\gamma_{2J}\omega(\alpha_{2J})\frac{\tau^{1-\alpha_{2J}}}{2^{1-\alpha_{2J}}\Gamma(2-\alpha_{2J})}\left\langle\mathfrak{A}\delta_t e_i^{k+1/2},\delta_t e_i^{k+1/2}\right\rangle=\frac{\omega(1)\Delta\alpha}{3}\left\langle\mathfrak{A}\delta_t e_i^{k+1/2},\delta_t e_i^{k+1/2}\right\rangle,
\end{aligned}$$

as $\alpha_{2J}=1$ and $\gamma_{2J}=\frac{1}{3}$. Furthermore, using the definition of $\mathfrak{A}$, we have that

$$\left\langle\mathfrak{A}\delta_t e_i^{k+1/2},\delta_t e_i^{k+1/2}\right\rangle=\left\langle\frac{1}{12}\left(\delta_t e_{i-1}^{k+1/2}+10\delta_t e_i^{k+1/2}+\delta_t e_{i+1}^{k+1/2}\right),\delta_t e_i^{k+1/2}\right\rangle\geq\frac{5}{6}\left\langle\delta_t e_i^{k+1/2},\delta_t e_i^{k+1/2}\right\rangle,$$

which means that the l.h.s. of (21) is bounded below by

$$\begin{aligned}
&\Delta\alpha\sum_{l=0}^{2J}\gamma_l\omega(\alpha_l)\left\langle\mathfrak{A}\left[\omega_1^l e_i^k+\sum_{m=1}^{k-1}(\omega_{k-m+1}^l-\omega_{k-m}^l)e_i^m-\omega_k^l e_i^0+\frac{\sigma^l}{2^{1-\alpha_l}}\left(e_i^{k+1}-e_i^k\right)\right],\delta_t e_i^{k+1/2}\right\rangle\\
&\geq\frac{5\omega(1)\Delta\alpha}{18}\left\|\delta_t e_i^{k+1/2}\right\|^2.
\end{aligned}\tag{34a}$$

Substituting Eqs. (30)–(32) and (34a) in (21), we obtain

$$\begin{aligned}
\frac{K}{2\tau}(|e_i^{k+1}|_1^2-|e_i^k|_1^2)\leq{}&\frac{9}{\Delta\alpha\omega(1)}c_1^2\left(\|e_i^k\|^2+\|e_i^{k-1}\|^2\right)+\frac{9}{5\Delta\alpha\omega(1)}c_2^2\left(\|e_i^{k+1-n}\|^2+\|e_i^{k-n}\|^2\right)\\
&+\frac{18}{10\Delta\alpha\omega(1)}Lc_3^2\left(\tau+h^4+\Delta\alpha^4\right)^2,
\end{aligned}\tag{34b}$$

multiply both sides by $\frac{2\tau}{K}$ and summing up for $k$ noticing (20), we get

$$|e_i^{k+1}|_1^2\leq\frac{9}{K\Delta\alpha\omega(1)}\left(c_1^2+\frac{c_2^2}{5}\right)\tau\sum_{m=1}^{k}\|e^m\|^2+\frac{18k\tau}{10K\Delta\alpha\omega(1)}Lc_3^2\left(\tau+h^4+\Delta\alpha^4\right)^2,\quad 0\leq k\leq\ell. \tag{34c}$$

Eq. (34c) is ready to apply Gronwall inequality to obtain:

$$|e^{\ell+1}|_1^2\leq\frac{9TLc_3^2}{5K\Delta\alpha\omega(1)}\exp\left(\frac{3L^2\left(c_1^2+\frac{c_2^2}{5}\right)T}{2K\Delta\alpha\omega(1)}\right)\left(\tau+h^4+\Delta\alpha^4\right)^2, \tag{34d}$$

using Eq. (15), we obtain

$$\|e^{\ell+1}\|_\infty\leq\frac{\sqrt{L}}{2}|e^{\ell+1}|_1\leq\frac{3c_3L}{2}\sqrt{\frac{T}{5K\Delta\alpha\omega(1)}}\exp\left(\frac{3L^2\left(c_1^2+\frac{c_2^2}{5}\right)T}{4K\Delta\alpha\omega(1)}\right)\left(\tau+h^4+\Delta\alpha^4\right). \tag{34e}$$

So, the inductive step is achieved and this completes the proof. □

To discuss the stability of the difference scheme (14a)–(14c), we also use the discrete energy method in the same way like the discussion of the convergence.

Let $\{v_i^k \mid 0 \le i \le M, 0 \le k \le N\}$ be the solution of

$$\Delta\alpha \sum_{l=0}^{2J} \gamma_l \omega(\alpha_l)\mathfrak{A}\left[\omega_1 v^k + \sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m})v^m - \omega_k v^0 + \sigma \frac{(v_i^{k+1} - v_i^k)}{2^{1-\alpha_l}}\right]$$
$$= K\delta_x^2 v_i^{k+1/2} + \mathfrak{A} f\left(x_i, t_{k+1/2}, \frac{3}{2}v_i^k - \frac{1}{2}v_i^{k-1}, \frac{1}{2}v_i^{k+1-n} + \frac{1}{2}v_i^{k-n}\right), \tag{35}$$

$$v_0^k = \phi_0(t_k), \qquad v_M^k = \phi_L(t_k), \quad 1 \le k \le N, \tag{36}$$

$$v_i^k = \psi(x_i, t_k) + \rho_i^k, \quad 0 \le i \le M, \quad -n \le k \le 0, \tag{37}$$

where $\rho_i^k$ is the perturbation of $\psi(x_i, t_k)$.

Following the same steps as in the proof of convergence theorem, the following result is obtained.

**Theorem 3** (*Stability Theorem*). *Let $\theta_i^k = v_i^k - u_i^k$, $0 \le i \le M$, $-n \le k \le N$, and the existence of constants $c_4, c_5, h_0, \tau_0$ which fulfill*

$$\|\theta^k\|_\infty \le c_4\sqrt{\tau}\sum_{k=-n}^{0}\|\rho^k\|, \quad 0 \le k \le N, \ \|\rho^k\| = \sqrt{h\sum_{i=1}^{M-1}(\phi_i^k)^2},$$

*conditioned by*

$$h \le h_0, \ \tau \le \tau_0, \quad \max_{\substack{-n\le k\le 0\\ 0\le i\le M}} |\rho_i^k| \le c_5.$$

## 4. Numerical experiments

Let $u_i^k = u(k\tau, h, \Delta\alpha)$ be the solution of the constructed difference scheme (14a)–(14c) with the step sizes $\tau, h, \Delta\alpha$. Define the maximum norm error by $E(\tau, h, \Delta\alpha) = \max_{\substack{0\le i\le M\\ 0\le k\le N}} \|U_i^k - u_i^k\|_\infty$.

Define the following error rates, $rate_1 = \log_2\left(\frac{E(2\tau,h,\Delta\alpha)}{E(\tau,h,\Delta\alpha)}\right)$, $rate_2 = \log_2\left(\frac{E(16\tau,2h,2\Delta\alpha)}{E(\tau,h,\Delta\alpha)}\right)$. We just adapt some numerical examples which appeared in [21].

**Example 1.**

$$\int_0^1 \Gamma(3-\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}d\alpha = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t),u(x,t-s)), \quad t \in (0,1), \ 0 < x < 2, \tag{38}$$

$$f(x,t,u(x,t),u(x,t-s)) = \frac{6(t^2-t)(2-x)x}{\ln(t)} - u(x,t-s) + 2t^2 + x(2-x)(t-s)^2,$$

with the following initial and boundary conditions

$$u(x,t) = t^2(2x - x^2), \quad 0 \le x \le 2, \ t \in [-s, 0), \ s > 0, \tag{39}$$
$$u(0,t) = u(2,t) = 0, \quad t \in [0,1]. \tag{40}$$

The analytical solution of this problem is

$$u(x,t) = t^2(2x - x^2). \tag{41}$$

**Example 2.**

$$\int_0^1 \Gamma\left(\frac{7}{2}-\alpha\right)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}d\alpha = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t),u(x,t-s)), \quad t \in (0,1), \ 0 < x < 1, \tag{42}$$

$$f(x,t,u(x,t),u(x,t-s)) = \frac{15\sqrt{\pi}(t-1)t^{\frac{3}{2}}(x-1)x}{8\ln(t)} + u^2(x,t-s) - 2t^{\frac{5}{2}} - (x^2-x)^2(t-s)^5,$$

with the following initial and boundary conditions

$$u(x,t) = t^{\frac{5}{2}}(x - x^2), \quad 0 \le x \le 1, \ t \in [-s, 0), \ s > 0, \tag{43}$$
$$u(0,t) = u(1,t) = 0, \quad t \in [0,1]. \tag{44}$$

**Table 1.1**
Errors and convergence orders of the difference scheme (14a)–(14c) in time variable with $h = 1/300$ and $\Delta\alpha = 1/300$.

| $\tau$ | $E(\tau, h, \Delta\alpha)$ | $rate_1$ |
|---|---|---|
| $\frac{1}{10}$ | 0.00135 | |
| $\frac{1}{20}$ | 0.00068 | 0.9784 |
| $\frac{1}{40}$ | 0.000344 | 0.9825 |
| $\frac{1}{80}$ | 0.00017 | 0.9876 |
| $\frac{1}{160}$ | 0.000087 | 0.9935 |
| $\frac{1}{320}$ | 0.000044 | 0.9978 |
| $\frac{1}{640}$ | 0.0000218 | 0.9996 |

**Table 1.2**
Errors and convergence orders of the difference scheme (14a)–(14c) with an optimal step size ratio.

| $\tau$ | $h$ | $\Delta\alpha$ | $E(\tau, h, \Delta\alpha)$ | $rate_2$ |
|---|---|---|---|---|
| $\frac{1}{2(4^3)}$ | $\frac{2}{8}$ | $\frac{1}{4}$ | 0.000216 | |
| $\frac{1}{4(8^3)}$ | $\frac{2}{16}$ | $\frac{1}{8}$ | 0.0000130754 | 4.0521 |
| $\frac{1}{8(16^3)}$ | $\frac{2}{32}$ | $\frac{1}{16}$ | $8.1371 \times 10^{-7}$ | 4.0062 |

**Table 2.1**
Errors and convergence orders of the difference scheme (14a)–(14c) in time variable with $h = 1/500$ and $\Delta\alpha = 1/400$.

| $\tau$ | $E(\tau, h, \Delta\alpha)$ | $rate_1$ |
|---|---|---|
| $\frac{1}{10}$ | 0.00123 | |
| $\frac{1}{20}$ | 0.00062 | 0.9853 |
| $\frac{1}{40}$ | 0.0003 | 0.9931 |
| $\frac{1}{80}$ | 0.00016 | 0.9952 |
| $\frac{1}{160}$ | 0.000078 | 0.9976 |
| $\frac{1}{320}$ | 0.000039 | 0.9985 |

**Table 2.2**
Errors and convergence orders of the difference scheme (14a)–(14c) with an optimal step size ratio.

| $\tau$ | $h$ | $\Delta\alpha$ | $E(\tau, h, \Delta\alpha)$ | $rate_2$ |
|---|---|---|---|---|
| $\frac{1}{6^4}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | 0.000135 | |
| $\frac{1}{12^4}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $8.69397 \times 10^{-6}$ | 3.9568 |
| $\frac{1}{24^4}$ | $\frac{1}{24}$ | $\frac{1}{24}$ | $5.48064 \times 10^{-7}$ | 3.9876 |

The analytical solution of this problem is

$$u(x, t) = t^{\frac{5}{2}}(x^2 - x). \tag{45}$$

In Tables 1.1 and 2.1, the error rates of the numerical solutions for test Examples 1 and 2 are computed with different time steps and fixed, sufficiently small $h$ and $\Delta\alpha$ and with delay parameters $s = 1$, $s = 0.5$ respectively. The approximation of time-fractional derivatives dominates the computational errors in view of the sufficiently small step sizes in space and distributed-order variables. From these tables, one can conclude that the convergence order in time is one in maximum norm. The computational results are in accordance with the theoretical results. In Tables 1.2 and 2.2, the computational results are displayed with an optimal step size ratio, i.e. $N = M^4 = (2J)^4$. One can conclude from these tables that reducing the spatial step size h and the distributed-order step size $\Delta\alpha$ by a factor of 2 corresponds to the decreasing of computational errors in discrete maximum norm by a factor of 16 gives fourth order convergence approximately.

## 5. Conclusions

The major aim of this work which lies in building a linearized difference scheme to solve a class of distributed order fractional diffusion equations with nonlinear delay is achieved. The stability and convergence analyses for the numerical solution are discussed. The proposed numerical test examples supported our theoretical results. The resulted difference scheme can be easily applied for two dimensional delay problems with distributed orders. For future consideration, we will increase the time convergence order to two by predicting another approximation for time Caputo fractional derivative at $t_{k+1/2}$ with convergence order $3 - \alpha$.

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## References

[1] M. Raberto, E. Scalas, F. Mainardi, Waiting-times returns in high frequency financial data: an empirical study, Physica A 314 (2002) 749–755.
[2] X. Li, M. Xu, X. Jiang, Homotopy perturbation method to time- fractional diffusion equation with a moving boundary conditions, Appl. Math. Comput. 208 (2009) 434–439.
[3] R.L. Bagley, P.J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, J. Rheol. 27 (1983) 201–210.
[4] J.A.T. Machado, Discrete-time fractional-order controllers, Fract. Calc. Appl. Anal. 4 (2001) 47–66.
[5] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional Calculus Models and Numerical Methods, in: Series on Complexity Non linearity and Chaos, World Scientific, Boston, 2012.
[6] W. Schneider, W. Wyss, Fractional diffusion and wave equations, J. Math. Phys. 30 (1989) 134–144.
[7] W. Wyss, The fractional diffusion equation, J. Math. Phys. 27 (1986) 27–82.
[8] M. Caputo, Distributed order differential equations modelling dielectric induction and diffusion, Fract. Calc. Appl. Anal. 4 (2001) 421–442.
[9] T.T. Hartley, C.F. Lorenzo, Fractional System Identification: An Approach Using Continuous Order-distributions, NASA Tech. Memo, 1999.
[10] T.M. Atanackovic, A generalized model for the uniaxial isothermal deformation of a viscoelastic body, Acta Mech. 159 (2002) 77–86.
[11] Ninghu Su, Paul N. Nelson, Sarah Connor, The distributed-order fractional diffusion-wave equation of groundwater flow: Theory and application to pumping and slug tests, J. Hydrol. 529 (2015) 1262–1273.
[12] T.M. Atanackovic, L. Opranica, S. Pilipovic, On a nonlinear distributed order fractional differential equation, J. Math. Anal. Appl. 328 (2007) 590–608.
[13] T.M. Atanackovic, M. Budincevic, S. Pilipovic, On a fractional distributed-order oscillator, J. Phys. A: Math. Gen. 38 (2005) 6703–6713.
[14] T.M. Atanackovic, S. Pilipovic, D. Zorica, Distributed-order fractional wave equation on a finite domain. Stress relaxation in a rod, Internat. J. Engrg. Sci. 49 (2011) 175–190.
[15] R. Gorenflo, Y. Luchko, M. Stojanovic, Fundamental solution of a distributed order time fractional diffusion wave equation as a probability density, Fract. Calc. Appl. Anal. 16 (2) (2013) 297–316.
[16] M.M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection–dispersion flow equations, J. Comput. Appl. Math. 172 (2004) 65–77.
[17] J. Ren, Z.Z. Sun, Maximum norm error analysis of difference schemes for fractional diffusion equations, Appl. Math. Comput. 256 (2015) 299–314.
[18] F. Huang, A time–space collocation spectral approximation for a class of time fractional differential equations, Int. J. Differ. Equ. 2012 (2012) 19. Article ID 495202.
[19] M.M. Khader, Talaat.S. El Danaf, A.S. Hendy, A computational matrix method for solving systems of high order fractional differential equations, Appl. Math. Model. 37 (2013) 4035–4050.
[20] F. Mainardi, G. Pagnini, A. Mura, R. Gorenflo, Time-fractional diffusion of distributed order, J. Vib. Control 14 (2008) 1267–1290.
[21] M.L. Morgado, M. Rebelo, Numerical approximation of distributed order reaction–diffusion equations, J. Comput. Appl. Math. 275 (2015) 216–227.
[22] Kai. Dithelm, Nevile J. Ford, Numerical analysis for distributed order differential equations, J. Comput. Appl. Math. 225 (2009) 96–104.
[23] Nevile J. Ford, M.L. Morgado, Distributed order equations as boundary value problems, Comput. Math. Appl. 64 (2012) 2973–2981.
[24] J.T. Katsikadelis, Numerical solution of distributed order fractional differential equations, J. Comput. Phys. 259 (2014) 11–22.
[25] Guang-hau Gao, Zhi-zhong Sun, Two alternating direction implicit difference schemes with the extrapolation method for the two-dimensional distributed-order differential equations, Comput. Math. Appl. 69 (2015) 926–984.
[26] Guang-hau Gao, Hai-wei Sun, Zhi-zhong Sun, Some high order difference schemes for distributed-order differential equations, J. Comput. Phys. 289 (2015) 337–359.
[27] M. Dehghan, R. Salehi, Solution of a nonlinear time-delay model in biology via semi-analytical approaches, Comput. Phys. Comm. 181 (2010) 1255–1265.
[28] J. Tumwiine, S. Luckhaus, J.Y.T. Mugisha, L.S. Luboobi, An age-structured mathematical medol for the within host dynamics of malaria and the immune system, J. Math. Model. Algorithms 7 (2008) 79–97.
[29] A. Bellen, M. Zennaro, Numerical Methods for Delay Differential Equations, Oxford university press, 2003.
[30] Q. Zhang, C. Zhang, A new linearized compact multisplitting scheme for the nonlinear convection-reaction–diffusion equations with delay, Commun. Nonlinear Sci. Numer. Simul. 18 (2013) 3278–3288.
[31] A. Ashyraliev, P.E. Sobolevskii, Well-posedness of parabolic difference equations, in: Operator Theory Advances and Applications, Vol. 69, Birkhäuser, 1994.
[32] F. Colombo, D. Guidetti, A global in time existence and uniqueness result for a semilinear integro-differential parabolic inverse problem in Sobolev spaces, Math. Models Methods Appl. Sci. 17 (2007) 537–565.
[33] R.H. De Staelen, M. Slodicka, Reconstruction of a convolution kernel in a semilinear parabolic problem based on a global measurement, Nonlinear Anal. TMA 112 (2015) 43–57.
[34] Andrey Lekomtsev, Vladimir Pimenov, Convergence of the scheme with weights for the numerical solution of a heat conduction equation with delay for the case of variable coefficient of heat conductivity, Appl. Math. Comput. 256 (2015) 83–93.
[35] V.G. Pimenov, A.S. Hendy, Numerical studies for fractional functional differential equations with delay based on BDF-Type shifted Chebyshev polynomials, Abstr. Appl. Anal. 2015 (2015) article ID 510875.
[36] J.A. Ferreira, Energy estimates for delay diffusion–reaction equations, J. Comput. Math. 26 (4) (2008) 536–553.
[37] Z.B. Zhang, Z.Z. Sun, A linearized compact difference scheme for a class of nonlinear delay partial differential equations, Appl. Math. Model. 37 (2013) 742–752.
[38] I. Karatay, N. Kale, S.R. Bayramoglu, A new difference scheme for time fractional heat equations based on Crank-Nichlson method, Fract. Calc. Appl. Anal. 16 (2013) 893–910.